

## Stability of fast parallel MHD shock waves in polytropic gas

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**ABSTRACT.** – The stability of plane shock waves in Magnetohydrodynamics for an ideal medium is studied. Stability results are obtained for the special case of fast parallel shock waves in a polytropic gas. Linear stability is proved for a polytropic gas with arbitrary  $\gamma$ . The domain of structural (nonlinear) stability, where the uniform Lopatinsky condition is fulfilled for the stability problem, is found. © Elsevier, Paris.

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### 1. Introduction

It is known that the equations of Magnetohydrodynamics (MHD) for an ideal medium tolerate the presence of strong discontinuities (for example, shock waves). In this connection, the question of the stability of shock waves in MHD with respect to small perturbations is of great interest. On the other hand, it should be noted that unlike the situation in Gas Dynamics (see D'yakov, 1954; Erpenbeck, 1962; Blokhin, 1981, 1982, 1986; Majda, 83a,b, 1984) the stability problem for MHD shock waves has not been fully investigated. After the publication of the classical works of Hoffman and Teller (1950), Akhiezer et al. (1958) and Gardner and Kruskal (1964) only a few studies of the stability of MHD shock waves have been published.

The stability problem for MHD shock waves with respect to one-dimensional perturbations was studied by Akhiezer et al. (1958). It is known that there exist two types of evolutionary (Landau and Lifshitz, 1982) MHD shock waves: fast and slow shock waves (see, for example, Hoffman and Teller, 1950 or Kulikovskii and Lyubimov, 1962). Gardner and Kruskal (1964) have derived a condition for stability of the fast MHD shock wave for the special cases of parallel and transverse plane shock waves (the magnetic field is parallel or perpendicular to the normal to the front of a shock wave). Stability was thereby proved for a polytropic gas with  $\gamma < 3$ .

In the work of Lessen and Deshpande (1967) the stability of fast and slow plane MHD shock waves in the polytropic gas was investigated numerically. They show that the slow shock wave can be unstable. An analogous, but more complete, investigation has been carried out by Fillipova (1991) who found some instability domains also for fast MHD shock waves.

Blokhin and Druzhinin (1989a,b) (see also Blokhin, 1993), Blokhin and Trakhinin (1994) investigate fast and slow MHD plane shock waves in a polytropic gas in the asymptotic cases of strong and weak magnetic fields. The stability of fast shock waves in a weak magnetic field and the instability of slow shock waves in a strong magnetic field are proved. Note that in these works the stability of MHD shock waves is investigated with the help of a so-called “equational” approach. This approach to the investigation of strong discontinuities was first structurally formulated by Blokhin (1986) and implies the investigation of the well-posedness of linear initial boundary value (linear mixed) stability problems by means of dissipative energy integral techniques. If

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the stability problem is well-posed, the strong discontinuity is stable. Otherwise, it is unstable. The basic steps of the formulated “equational” approach are the following:

- symmetrization of the initial quasilinear system of hyperbolic conservation laws,
- linearization of the initial quasilinear equations and relations on the strong discontinuity surface,
- separation of ill-posedness domains for the formulated linear mixed stability problem (Hadamard example),
- construction of *a priori* estimates without loss of smoothness for the linear mixed stability problem with the help of dissipative energy integral techniques in such domains when the Hadamard example is not constructed.

If we succeeded in realizing the “equational” approach in full, then we have a rigorous mathematical basis for the linearization method as applied to the investigation of strong discontinuity stability. It is very important that the *a priori* estimates obtained are *without loss of smoothness* because in this case one can show that a so-called *uniform Lopatinsky condition* (see Kreiss, 1970; Majda, 1983a, 1984; Blokhin, 1986 and Sect. 3 of the present article) holds, and we can in principle carry our result forward to the initial level of a quasilinear system of hyperbolic conservation laws with boundary conditions on a curvilinear surface of strong discontinuity. For example, this was done by Blokhin (1981, 1982) for gas dynamic shock waves, and the local theorem of existence and uniqueness of the classical solution for the quasilinear Gas Dynamics system behind the curvilinear shock wave was proved. Note that the presence of an *a priori* estimate *with loss of smoothness* (this is the case when only the Lopatinsky condition holds and the uniform Lopatinsky condition is not fulfilled) does not allow the use of the main results obtained for the case of constant coefficients to the case of variable coefficients (and, so, to a quasilinear case). In this connection, it is important that the uniform Lopatinsky condition for the linear mixed stability problem is fulfilled.

Following Majda (1983a, 1984) we will say that a strong discontinuity is *structurally stable* if the boundary conditions of the corresponding linear stability problem satisfy the uniform Lopatinsky condition. Actually, the whole domain of parameters of the linear stability problem consists of the following subdomains:

- I. The domain, where the Lopatinsky condition is violated (*instability*);
- II. The domain of fulfilment of the uniform Lopatinsky condition (*structural stability*);
- III. The domain, where only the Lopatinsky condition holds, and the uniform Lopatinsky condition is not fulfilled.

As we have noted above, the fulfilment of the Lopatinsky condition (in the domains II and III) does not always guarantee the local existence of shock front solutions of a quasilinear system of conservation laws. In Sect. 3 of the present article we give some additional arguments which show that in the domain III it is impossible to be certain of the existence of a strong discontinuity (as a physical structure) at the linear level of investigation. In this connection, the union of domains II and III is the domain of *linear stability*. In fact, it is a stability domain in the ordinary sense that initial small perturbations do not grow exponentially with time, i.e., an ill-posed example of Hadamard type cannot be constructed for the linear stability problem. This type of stability is proved, for example, in the work of Gardner and Kruskal (1964) cited above.

We note that there is no general result (for arbitrary hyperbolic systems of conservation laws) analogous to that obtained by Blokhin (1981, 1982) (see also Majda, 1983b) for gas dynamic shock waves. On the other hand, in domain II an ill-posed example of Hadamard type cannot be constructed for the stability problem or for all similar problems which are obtained by a perturbation of the system and the boundary conditions. So, with a certain degree of strictness we can say that domain II is the domain of *structural – nonlinear – stability*. We note also that the structural stability of MHD shock waves is studied precisely in the above mentioned works of Blokhin (1993), Blokhin and Trakhinin (1994).

In the present work the structural stability of fast parallel MHD shock waves in a polytropic gas as tested by fulfilment of the uniform Lopatinsky condition is investigated. As was noted above, such a stability problem

was studied by Gardner and Kruskal (1964), and the linear stability of fast parallel MHD shock waves (as well transverse ones) is proved for a polytropic gas with  $\gamma < 3$ . In the present work the linear stability of fast parallel shock waves is shown for a polytropic gas with arbitrary  $\gamma$  ( $\gamma > 1$ ), and the condition for structural (nonlinear) stability is found.

## 2. Mathematical statement of the stability problem for a fast parallel MHD shock wave

The MHD equations for an ideal medium are written in the form of the following system of conservation laws (see, for example, Gardner and Kruskal, 1964; Kulikovskii and Lyubimov, 1962; Landau and Lifshitz, 1982):

$$(1) \quad \begin{aligned} \rho_t + \operatorname{div}(\rho \mathbf{v}) &= 0, \\ (\rho v_i)_t + \sum_{k=1}^3 (\Pi_{ik})_{x_k} &= 0, \quad i = 1, 2, 3, \\ \mathbf{H}_t - \operatorname{rot}(\mathbf{v} \times \mathbf{H}) &= 0, \\ (\mathcal{E}_0)_t + \operatorname{div} \mathcal{E} &= 0. \end{aligned}$$

Here  $\rho$  is the density of the gas;  $\mathbf{v} = (v_1, v_2, v_3)^*$ , the velocity of the gas (asterisk stands for transposition);  $t$ , the time;  $\mathbf{x} = (x_1, x_2, x_3)$ , the Cartesian coordinates;

$$\Pi_{ik} = \rho v_i v_k + p \delta_{ik} - \frac{1}{4\pi} \left( H_i H_k - \frac{1}{2} |\mathbf{H}|^2 \delta_{ik} \right) \quad (i, k = 1, 2, 3),$$

the components of the momentum flux;  $p$ , the pressure;  $\mathbf{H} = (H_1, H_2, H_3)^*$ , the magnetic field;

$$\mathcal{E}_0 = \rho E + \rho \frac{|\mathbf{v}|^2}{2} + \frac{|\mathbf{H}|^2}{8\pi}, \quad \mathcal{E} = \rho \mathbf{v} \left( E + \frac{|\mathbf{v}|^2}{2} + pV \right) + \frac{1}{4\pi} \mathbf{H} \times (\mathbf{v} \times \mathbf{H});$$

$E$  is the internal energy;  $V = 1/\rho$ .

The temperature  $T$  and the entropy  $S$  satisfy, as in Gas Dynamics, the thermodynamic identity

$$T dS = dE + p dV.$$

Thus, if we append the *state equation* of medium  $E = E(\rho, S)$  to system (1), then we close system (1), and can regard it as a system for finding the components of the vector  $\mathbf{U} = (p, S, \mathbf{v}^*, \mathbf{H}^*)^*$ .

System (1) should be supplemented by the condition

$$\operatorname{div} \mathbf{H} = 0,$$

which is, in fact, an additional requirement on the initial data for system (1).

Finally, system (1) implies the *additional conservation law* (entropy conservation)

$$(2) \quad (\rho S)_t + \operatorname{div}(\rho S \mathbf{v}) = 0,$$

which holds on smooth solutions. Note that law (2) was used by Godunov (1972) for the symmetrization of the MHD system (1). Following Blokhin (1993) we can rewrite system (1) in the symmetric form

$$(3) \quad A_0(\mathbf{U}) \mathbf{U}_t + \sum_{k=1}^3 A_k(\mathbf{U}) \mathbf{U}_{x_k} = 0,$$

where  $A_\alpha (\alpha = \overline{0,3})$  are symmetric matrices. Moreover, the diagonal matrix  $A_0$ ,

$$A_0 = \text{diag} \left( \frac{1}{\rho c^2}, 1, \rho, \rho, \rho, \frac{1}{4\pi}, \frac{1}{4\pi}, \frac{1}{4\pi} \right),$$

is positive definite, i.e., system (3) is symmetric  $t$ -hyperbolic (in the sense of Friedrichs, 1974). Here  $c = ((\rho^2 E_\rho)_\rho)^{1/2}$  is the speed of sound in the gas, and the matrices  $A_k$  ( $k = 1, 2, 3$ ) are described in detail in the monograph of Blokhin (1993).

Now we consider piecewise smooth solutions to system (1) with smooth parts separated by the surface of strong discontinuity with the equation

$$\tilde{f}(t, \mathbf{x}) = f(t, \mathbf{x}') - x_1 = 0$$

( $\mathbf{x}' = (x_2, x_3)$ ). The following jump conditions are satisfied on the surface of the strong discontinuity (see, for example, Kulikovskii and Lyubimov, 1962):

$$(4) \quad \begin{aligned} [j] &= 0, \quad [H_N] = 0, \quad \left[ jv_N + p + \frac{|\mathbf{H}|^2}{8\pi} \right] = 0, \\ j[v_{\tau_i}] &= \frac{H_N}{4\pi} [H_{\tau_i}], \quad H_N[v_{\tau_i}] = j[VH_{\tau_i}] \quad (i = 1, 2), \\ \left[ V\mathcal{E}_0 j + \left( p + \frac{|\mathbf{H}|^2}{8\pi} \right) v_N - \frac{H_N}{4\pi} (\mathbf{H}, \mathbf{v}) \right] &= 0. \end{aligned}$$

We use the conventional notations  $[F] = F^+ - F^-$  for every regularly discontinuous function  $F$ . The subscripts  $+$ ,  $-$  denote the value of the function ahead ( $\tilde{f} \rightarrow +0$ ) and behind ( $\tilde{f} \rightarrow -0$ ) the discontinuity front. Below we will write  $F$  instead of  $F^+$ , and  $F_\infty$  instead of  $F^-$ . Here

$$j = \rho(v_N - D_N), \quad v_N = (\mathbf{v}, \mathbf{N}), \quad H_N = (\mathbf{H}, \mathbf{N}), \quad v_{\tau_i} = (\mathbf{v}, \boldsymbol{\tau}_i), \quad i = 1, 2;$$

$\mathbf{N} = (1/|\nabla \tilde{f}|)(-1, f_{x_2}, f_{x_3})^*$  is the unit normal to the discontinuity front,  $D_N = -f_t/|\nabla \tilde{f}|$  is the speed of the discontinuity front in the normal direction;

$$|\nabla \tilde{f}| = (1 + f_{x_2}^2 + f_{x_3}^2)^{1/2}, \quad \boldsymbol{\tau}_1 = (f_{x_2}, 1, 0)^*, \quad \boldsymbol{\tau}_2 = (f_{x_3}, 0, 1)^*.$$

Let a strong discontinuity be a shock wave, i.e.,  $j \neq 0$ ,  $[\rho] \neq 0$ . We consider a plane stepshock with the equation  $x_1 = 0$  and the piecewise constant solution to system (1)

$$(5) \quad \mathbf{U}(t, \mathbf{x}) = \begin{cases} \hat{\mathbf{U}}_\infty = (\hat{p}_\infty, \hat{S}_\infty, \hat{v}_{1\infty}, \hat{v}_{2\infty}, \hat{v}_{3\infty}, \hat{H}_{1\infty}, \hat{H}_{2\infty}, \hat{H}_{3\infty})^*, & x_1 < 0; \\ \hat{\mathbf{U}} = (\hat{p}, \hat{S}, \hat{v}_1, \hat{v}_2, \hat{v}_3, \hat{H}_1, \hat{H}_2, \hat{H}_3)^*, & x_1 > 0; \end{cases}$$

which satisfies jump conditions (4) on the plane  $x_1 = 0$ . Here  $\hat{p}, \hat{p}_\infty, \hat{p}, \hat{p}_\infty, \hat{S}, \hat{S}_\infty, \hat{v}_k, \hat{v}_{k\infty}, \hat{H}_k, \hat{H}_{k\infty}$  ( $k = 1, 2, 3$ ) are constants. Let the piecewise constant solution (5) correspond to the case of fast shock waves, i.e., the fulfilment of the inequalities

$$(6a) \quad \hat{v}_{1\infty} > \hat{c}_{M\infty}^+, \quad \hat{c}_A < \hat{v}_1 < \hat{c}_M^+$$

is assumed with *compressibility conditions* (see, for example, Kulikovskii and Lyubimov, 1962)

$$(6b) \quad \hat{p} > \hat{p}_\infty, \quad \hat{\rho} > \hat{\rho}_\infty, \quad \hat{S} > \hat{S}_\infty, \quad \hat{v}_{1\infty} > \hat{v}_1 > 0.$$

Here  $\hat{c}_A = \hat{c}h_1$  is the Alfvén speed,  $\hat{c} = ((\rho^2 E_\rho)_\rho(\hat{\rho}, \hat{S}))^{1/2}$ ,  $h_k = \hat{H}_k/(\hat{c}\sqrt{4\pi\hat{p}})$  ( $k = 1, 2, 3$ ),  $\mathbf{h} = (h_1, h_2, h_3)^*$ ,  $q = |\mathbf{h}|$ ,  $h_1 = h_{1\infty} > 0$ ;

$$\hat{c}_M^\pm = \hat{c}_M^\pm(\hat{\mathbf{U}}) = \frac{\hat{c}}{\sqrt{2}} \{1 + q^2 \pm ((1 + q^2)^2 - 4h_1^2)^{1/2}\}^{1/2},$$

the fast and slow magnetic speeds of sound;  $\hat{c}_{M\infty}^+ = \hat{c}_M^+(\hat{\mathbf{U}}_\infty)$ . For the case of a polytropic gas with the state equation

$$E = \frac{pV}{\gamma - 1},$$

where  $\gamma$  is the adiabatic exponent ( $\gamma > 1$ ), conditions (6) are equivalent to the inequalities (see, for example, Blokhin, 1993)

$$(7) \quad 0 < q < M, \quad \frac{\gamma - 1}{2\gamma} < M^2 < 1.$$

Here  $M = \hat{v}_1/\hat{c}$  is the Mach number.

We will assume the stepshock to be parallel, i.e.,

$$(8) \quad \hat{H}_2 = \hat{H}_{2\infty} = \hat{H}_3 = \hat{H}_{3\infty} = 0.$$

Then the following can be obtained from system (4):

$$[\hat{v}_2] = 0, \quad [\hat{v}_3] = 0.$$

Hence we can choose a reference frame in which

$$(9) \quad \hat{v}_2 = \hat{v}_{2\infty} = \hat{v}_3 = \hat{v}_{3\infty} = 0.$$

Linearizing system (1) with respect to solution (5) and taking into account (8), (9), we obtain the linear stability problem for fast parallel MHD shock waves which will determine the vector of small perturbations  $\delta\mathbf{U}$  (in order to simplify the notation we will again indicate the vector  $\delta\mathbf{U}$  by  $\mathbf{U}$ ) and the small displacement of the shock front  $F = \delta f$ . It is a mixed problem for linear systems in the half-spaces  $x_1 > 0$  and  $x_1 < 0$  with boundary conditions on the plane  $x_1 = 0$ . In view of the first inequality from (5), all the characteristics of the linearised MHD system (in the one-dimensional case) for  $x_1 < 0$  are absent (see, for example, Gardner and Kruskal, 1964), and, so, for  $x_1 < 0$  the solution is completely determined by the initial data given at  $t = 0$ . Thus, without loss of generality we can assume that  $\mathbf{U} \equiv 0$  for  $x_1 < 0$ . Moreover, as was noted by Gardner and Kruskal (1964), because of the symmetry of the stability problem with respect to the  $x_{2,3}$ -dependence of perturbations for parallel MHD shocks we can consider only two-dimensional perturbations.

Taking account of the arguments above and following Blokhin (1993), the above mentioned stability problem for a fast parallel MHD shock wave in a polytropic gas takes the following form of a linear mixed problem (without loss of generality we consider the two-dimensional case  $\mathbf{x} = (x_1, x_2)$ ).

*Problem  $\mathcal{F}$ .* Find a solution to the system of equations

$$\begin{aligned}
 (10) \quad & Lp + \operatorname{div} \mathbf{v} = 0, \\
 & LS = 0, \\
 & M^2 Lv_1 + \frac{\partial p}{\partial x_1} = 0, \\
 & M^2 Lv_2 + \frac{\partial p}{\partial x_2} + q \frac{\partial H_1}{\partial x_2} - q \frac{\partial H_2}{\partial x_1} = 0, \\
 & LH_1 + q \frac{\partial v_2}{\partial x_2} = 0, \\
 & LH_2 - q \frac{\partial v_2}{\partial x_1} = 0
 \end{aligned}$$

for  $t > 0$ ,  $\mathbf{x} \in R_+^2$ , satisfying the boundary conditions

$$\begin{aligned}
 (11) \quad & v_1 = a_1 p, \quad \frac{\partial F}{\partial t} = a_2 p, \quad v_2 = a_3 \frac{\partial F}{\partial x_2}, \\
 & S = a_4 p, \quad H_2 = q v_2, \quad H_1 = 0
 \end{aligned}$$

for  $t > 0$ ,  $x_1 = 0$  and  $x_2 \in R$  and the initial data

$$(12) \quad \mathbf{U}(0, \mathbf{x}) = \mathbf{U}_0(\mathbf{x}), \quad \mathbf{x} \in R_+^2; \quad F(0, x_2) = F_0(x_2), \quad x_2 \in R$$

for  $t = 0$ . Here  $L = \frac{\partial}{\partial t} + \frac{\partial}{\partial x_1}$ ;  $R_+^2 = \{\mathbf{x} | x_1 > 0, x_2 \in R\}$ ,

$$\mathbf{U} = (p, S, \mathbf{v}^*, \mathbf{H}^*)^*, \quad \mathbf{v} = (v_1, v_2)^*, \quad \mathbf{H} = (H_1, H_2)^*;$$

$p, S, v_k, H_k$  ( $k = 1, 2$ ) stand for small perturbations of the pressure, entropy, components of the velocity vector and magnetic field reduced in the corresponding way to a dimensionless (see Blokhin, 1993);

$$\begin{aligned}
 a_1 &= \frac{3 - \gamma + (3\gamma - 1)M^2}{2M^2(2 + (\gamma - 1)M^2)}, \quad a_2 = -\frac{\gamma + 1}{4M^2}, \\
 a_3 &= \frac{2(1 - M^2)}{(\gamma + 1)(M^2 - q^2)}, \quad a_4 = \frac{(\gamma - 1)(1 - M^2)^2}{M^2(2 + (\gamma - 1)M^2)};
 \end{aligned}$$

$F$  is a small displacement of the shock front (with the equation  $x_1 = F(t, x_2)$ ).

*Remark 1.* – While solving the mixed Problem  $\mathcal{F}$ , we also determine the function  $F = F(t, x_2)$ . For this purpose, one of the boundary conditions (11) must be the equation for determination of the function  $F$ .

*Remark 2.* – Necessary conditions for the well-posedness of Problem  $\mathcal{F}$  as well for the stability of shock waves are the geometrical Lax conditions (Jeffrey, 1976), which assure that the problem is well formulated with respect to the number of boundary conditions at  $x_1 = 0$ . Shock solutions satisfying the Lax conditions are said to be evolutionary (Landau and Lifshitz, 1982). Mixed Problem  $\mathcal{F}$  satisfies the Lax conditions if inequalities (7) hold (this fact is shown in particular in Blokhin, (1993); see also, for example, Kulikovskii and Lyubimov, 1962), i.e., the fast parallel shock wave in this case is evolutionary.

### 3. Uniform Lopatinsky condition

In this section, following Kreiss (1970), we define a so-called *uniform Lopatinsky condition* for linear mixed problems in the form of Problem  $\mathcal{F}$  and, applying the ideas of Gardner and Kruskal (1964), give an equivalent definition of this condition for the case when only one characteristic of the system (in the one-dimensional case) is absent, and the others are present.

System (10) can be rewritten in the form of the following symmetric  $t$ -hyperbolic system:

$$(13) \quad B_0 \mathbf{U}_t + B_1 \mathbf{U}_{x_1} + B_2 \mathbf{U}_{x_2} = 0,$$

where  $B_0 = \text{diag}(1, 1, M^2, M^2, 1, 1)$  is the diagonal matrix:

$$B_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & M^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & M^2 & 0 & -q \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -q & 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Eliminating the function  $F(t, x_2)$  from (11) we obtain the following boundary conditions at  $x_1 = 0$  for system (13):

$$(14) \quad \begin{aligned} v_1 &= a_1 p, & \frac{\partial v_2}{\partial t} &= a_5 \frac{\partial p}{\partial x_2}, \\ S &= a_4 p, & H_2 &= q v_2, & H_1 &= 0, \end{aligned}$$

where  $a_5 = (M^2 - 1)/(2M^2(M^2 - q^2))$ .

Applying a Fourier-Laplace transform to system (13) and the boundary conditions (14) we obtain the following boundary value problem for the system of ordinary differential equations:

$$(15) \quad -\frac{d\hat{\mathbf{U}}}{dx_1} = \mathcal{M}(s, \omega)\hat{\mathbf{U}}, \quad x_1 > 0,$$

$$(16) \quad \mathcal{M}_0(s, \omega)\hat{\mathbf{U}} = 0, \quad x_1 = 0.$$

Here

$$\hat{\mathbf{U}} = \hat{\mathbf{U}}(x_1) = \frac{1}{2\pi} \int \int_{R^2} e^{-st - i\omega x_2} \mathbf{U}(t, x_1, x_2) dt dx_2$$

is the Fourier-Laplace transform of the vector function  $\mathbf{U}(t, \mathbf{x})$ ;

$$s = \eta + i\xi, \quad \eta > 0; \quad (\xi, \omega) \in R^2, \quad \mathcal{M} = \mathcal{M}(s, \omega) = -B_1^{-1}(sB_0 + i\omega B_2),$$

$$\mathcal{M}_0 = \mathcal{M}_0(s, \omega) = \begin{pmatrix} a_1 & 0 & -1 & 0 & 0 & 0 \\ a_4 & -1 & 0 & 0 & 0 & 0 \\ \omega a_5 & 0 & 0 & -s & 0 & 0 \\ 0 & 0 & 0 & q & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

In applying the Fourier-Laplace transform we, as usual, assume that

$$U(t, \mathbf{x}) = 0 \text{ for } t \leq 0.$$

Note that five eigenvalues of matrix  $B_0^{-1}B_1$  are positive and one is negative. It follows that for all real  $\omega$  and  $\eta > 0$  five eigenvalues  $\lambda$  of the matrix  $\mathcal{M}$  lie in the left semi-plane ( $\operatorname{Re}\lambda < 0$ ), and one eigenvalue lies in the right semi-plane ( $\operatorname{Re}\lambda > 0$ ). Indeed, this property of the eigenvalues  $\lambda$  is valid for  $\omega = 0$  and  $\eta > 0$ . On the other hand, as system (13) is symmetric  $t$ -hyperbolic then the assumption  $\operatorname{Re}\lambda = 0$  follows on  $\eta = 0$ . Hence the location of the eigenvalues  $\lambda$  relative to the imaginary axis of the complex  $\lambda$ -plane is independent of  $\omega$ . We note also that the proved property of eigenvalues of the matrix  $\mathcal{M}$  is the general fact for matrices in the form of  $\mathcal{M}$  as is proved in particular (in other terms) by Gardner and Kruskal (1964).

So, we reduce the matrix  $\mathcal{M}$  to the form

$$\mathcal{M} = \Lambda \begin{pmatrix} b & 0 \\ 0 & Q \end{pmatrix} \Lambda^{-1},$$

where  $\Lambda$  is a non-degenerate matrix;  $b$  is a number with  $\operatorname{Re} b > 0$ ; all the eigenvalues  $\lambda$  of the matrix  $Q$  lie in the left semi-plane ( $\operatorname{Re}\lambda < 0$ ). We seek the *bounded* solution of problem (15), (16) in the form:

$$\hat{U} = \Lambda \begin{pmatrix} 0 \\ e^{Qx_1} \mathbf{C} \end{pmatrix},$$

where  $\mathbf{C}$  is a constant vector which is found from the system

$$\mathcal{M}_0 \Lambda \begin{pmatrix} 0 \\ \mathbf{C} \end{pmatrix} = \mathcal{L}(\mathcal{M}_0, \eta, \xi, \omega) \mathbf{C} = 0.$$

If  $\det \mathcal{L}(\mathcal{M}_0, \eta, \xi, \omega) = 0$  for some  $\eta > 0$ ,  $(\xi, \omega) \in R^2$  then the sequence of vector functions

$$\mathbf{U}_k(t, \mathbf{x}) = e^{-\sqrt{k} + k(\eta t + i\xi t + i\omega x_2)} \Lambda(\eta, \xi, \omega) \begin{pmatrix} 0 \\ e^{kQ(\eta, \xi, \omega)x_1} \mathbf{C} \end{pmatrix}$$

( $k = 1, 2, 3, \dots$ ), which are the solutions of the mixed problem (13), (14) with special initial data, is the ill-posedness example of Hadamard type.

Thus, following Kreiss (1970), we say that boundary conditions (14) satisfy the *Lopatinsky condition* (LC) if  $\det \mathcal{L}(\mathcal{M}_0, \eta, \xi, \omega) \neq 0$  for all  $\eta > 0$ ,  $(\xi, \omega) \in R^2$ . Moreover, boundary conditions (14) satisfy the *uniform Lopatinsky condition* (ULC) if  $\det \mathcal{L}(\mathcal{M}_0, \eta, \xi, \omega) \neq 0$  for all  $\eta \geq 0$ ,  $(\xi, \omega) \in R^2 (\eta^2 + \xi^2 + \omega^2 \neq 0)$ .

In the work of Kreiss (1970) mixed problems with boundary conditions which do not contain derivatives of unknown functions (the matrix  $\mathcal{M}_0$  in this case is independent of  $s$  and  $\omega$ ) were considered. In domain II (see Sect. 1) *a priori*  $L_2$ -estimates *without loss of smoothness* are obtained for such mixed problems. In domain III (see Sect. 1) only *a priori* estimates *with loss of smoothness* are obtained.

Note that analogous results for mixed problems with boundary conditions including derivatives of unknown functions do not exist. For example, stability problems for strong discontinuities fall under this case. However, in domain III of the linear mixed problem of the stability of strong discontinuity (such cases are called as neutrally stable by physicists) we cannot be certain about the existence of a strong discontinuity (as a physical structure). For example, Blokhin (1986) has shown that for the linear mixed problem of the stability of gas dynamic shock waves in domain III (this domain is named by D'yakov (1954) as the domain of spontaneous sound radiation by the discontinuity) one can so perturb the system and the boundary conditions that an ill-posedness example of Hadamard type can be constructed for the perturbed mixed problem. Thus, so-called neutral "stability" can be found in practice as instability. For this reason, it is very important to verify the fulfilment of the ULC for stability problems for strong discontinuities.

Since only one eigenvalue of the matrix  $B_0^{-1}B_1$  is negative, and applying the ideas of Gardner and Kruskal (1964), we give now an equivalent definition of the ULC for Problem  $\mathcal{F}$ . So, according to Gardner and Kruskal (1964), we can also write out the solution of problem (15), (16) in the following form:

$$(17) \quad \hat{\mathbf{U}}(x_1) = \frac{1}{2\pi i} \oint_C (sB_0 + \lambda B_1 + i\omega B_2)^{-1} B_1 \hat{\mathbf{U}}_0 \exp(\lambda x_1) d\lambda,$$

where  $C$  is a contour large enough to enclose all the singularities of the integrand;  $\hat{\mathbf{U}}_0$  is a constant vector satisfying boundary conditions (16) ( $\mathcal{M}_0 \hat{\mathbf{U}}_0 = \mathcal{M}_0 \hat{\mathbf{U}}(0) = 0$ ). Note that the singularities of the integrand are the eigenvalues of the matrix  $\mathcal{M}$  and satisfy the equation

$$(18) \quad \det(sB_0 + \lambda B_1 + i\omega B_2) = 0.$$

It follows from (17) that  $\hat{\mathbf{U}}(x_1)$  is a sum of residues at the poles of the integrand. As is noted above, there is one eigenvalue  $\lambda$  with  $\operatorname{Re} \lambda > 0$ , i.e., for this  $\lambda \exp(\lambda x_1) \rightarrow +\infty$  as  $x_1 \rightarrow +\infty$ . Hence the residue at this value of  $\lambda$  must be zero. As is shown by Gardner and Kruskal (1964), this is the same as the statement that for a given real  $\omega$  there exist complex numbers  $s$  and  $\lambda$  with

$$\operatorname{Re} s = \eta > 0, \quad \operatorname{Re} \lambda > 0,$$

such that the homogeneous system

$$(19) \quad (sB_0 + \lambda B_1 + i\omega B_2) \mathbf{X} = 0,$$

$$(20) \quad \mathbf{X}^* B_1 \hat{\mathbf{U}}_0 = 0$$

has a nonzero solution  $\mathbf{X}$ . We recall that these values of  $s$ ,  $\lambda$  and  $\omega$  must satisfy Eq. (18). As  $\lambda$  with  $\operatorname{Re} \lambda > 0$  is a simple eigenvalue then we can choose five linearly independent equations from system (19). Adding Eq. (20) to these equations we rewrite system (19), (20) in the form

$$G(s, \lambda, \omega) \mathbf{X} = 0,$$

where  $G = G(s, \lambda, \omega)$  is a quadratic matrix. If  $\det G = g(\mathcal{M}_0, \eta, \xi, \omega, \lambda) = 0$  then the sequence of vector functions

$$\mathbf{U}_k(t, \mathbf{x}) = e^{-\sqrt{k} + k(\eta t + i\xi t + i\omega x_2)} \hat{\mathbf{U}}(x_1)$$

( $k = 1, 2, 3, \dots$ ) is the ill-posedness example of Hadamard type for the mixed problem (13), (14) with special initial data.

So, boundary conditions (14) satisfy the LC if  $g(\mathcal{M}_0, \eta, \xi, \omega, \lambda) \neq 0$  for all  $\eta > 0$ ,  $(\xi, \omega) \in R^2$  and for  $\lambda$  with  $\operatorname{Re} \lambda > 0$ , where  $\lambda$  is the solution of (18).

Note that requirement  $\eta = 0$  implies  $\operatorname{Re} \lambda = 0$  (system (13) is symmetric). Let  $\lambda = \lambda(\eta, \xi, \omega)$  with  $\operatorname{Re} \lambda > 0$  for  $\eta > 0$  be a solution of (18), and  $\lambda_0 = \lambda(0, \xi, \omega)$ . Thus, boundary conditions (14) satisfy the ULC if  $g(\mathcal{M}_0, \eta, \xi, \omega, \lambda) \neq 0$  for all  $\eta \geq 0$ ,  $(\xi, \omega) \in R^2$  ( $\eta^2 + \xi^2 + \omega^2 \neq 0$ ) and for  $\lambda$  with  $\operatorname{Re} \lambda \geq 0$ , where  $\lambda$  is the solution of (18), and  $\lambda = \lambda_0$  for  $\eta = 0$ .

#### 4. Linear stability of a fast parallel MHD shock wave for a polytropic gas with arbitrary $\gamma$

Eq. (18) can be written explicitly as follows:

$$(21) \quad \Omega^2 \{ M^2 \Omega^2 (M^2 \Omega^2 - \lambda^2 + \omega^2) + q^2 (\omega^2 - \lambda^2) (M^2 \Omega^2 - \lambda^2) \} = 0,$$

where  $\Omega = s + \lambda$ . Note that  $g(\mathcal{M}_0, \eta, \xi, \omega, \lambda) \neq 0$  for  $\omega = 0$  ( $\eta \geq 0, \xi \in R, \eta^2 + \xi^2 \neq 0, \operatorname{Re} \lambda > 0$ ). Indeed, if  $\omega = 0$  then Eq. (21) follows  $\eta \neq 0$  ( $\eta > 0$ ) because  $\operatorname{Re} \lambda > 0$ . But Problem  $\mathcal{F}$  is correctly posed in accord with the number of boundary conditions (see Remark 2 in Sec. 2), and an ill-posedness example (in the one-dimensional case, i.e., under  $\omega = 0$ ) of Hadamard type cannot be constructed. Thus, we can assume that  $\omega \neq 0$  (or without loss of generality  $\omega > 0$ ). Moreover, without loss of generality we can assume that in Eq. (21)  $\omega = 1$ .

As

$$(22) \quad \eta > 0, \quad \operatorname{Re} \lambda > 0$$

the second factor of Eq. (21) must be equal to zero. This can be written

$$(23) \quad 1 - (q^2 + 1)(\nu^2 + \mu^2) + q^2 \nu^2 (\nu^2 + \mu^2) = 0,$$

where  $\nu, \mu$  are defined by

$$\nu = \frac{\lambda}{M\Omega}, \quad \mu = -\frac{i}{M\Omega}.$$

Eq. (23) has two roots  $\mu_1$  and  $\mu_2$  ( $\mu_2 = -\mu_1$ ). We choose the root  $\mu$  of (23) in the following form:

$$(24) \quad \mu = \mu_1 = -\frac{1}{z} \left\{ \frac{(z^2 - q^2)(z^2 - 1)}{z^2 - \frac{q^2}{1+q^2}} \right\}^{1/2} \frac{1}{\sqrt{1+q^2}},$$

where

$$(25) \quad z = 1/\nu$$

and

$$\frac{q^2}{1+q^2} < q^2 < M^2 < 1.$$

It follows from (22) that

$$\operatorname{Im}(1/\mu) > 0, \quad \operatorname{Im}(\nu/\mu) > 0.$$

In view of (24), (25), the last conditions imply that the domain of  $z$  is the right half ( $\operatorname{Re} z > 0$ ) of the  $z$ -plane with two segments of the real axis removed: the segment from  $q^2/(1+q^2)$  to  $q^2$  and the segment from 1 to  $+\infty$ . Note that conditions (22) are not fulfilled for  $\mu = \mu_2$ . From system (19) we obtain the relations

$$(26) \quad \begin{aligned} p &= -M \mu \frac{z^2}{1-z^2} v_2^{(1)}, \quad v_1^{(1)} = -\frac{\mu z}{z^2-1} v_2^{(1)}, \\ H_1^{(1)} &= M \mu q v_2^{(1)}, \quad H_2^{(1)} = \frac{Mq}{z} v_2^{(1)}, \quad S^{(1)} = 0 \end{aligned}$$

for components of the vector  $\mathbf{X} = (p^{(1)}, S^{(1)}, v_1^{(1)}, v_2^{(1)}, H_1^{(1)}, H_2^{(1)})^*$ .

In view of (14), the components of the vector  $\hat{\mathbf{U}}_0 = (p^{(0)}, S^{(0)}, v_1^{(0)}, v_1^{(0)}, v_2^{(0)}, H_1^{(0)}, H_2^{(0)})^*$  are connected by the following relations:

$$\begin{aligned} v_1^{(0)} &= a_1 p^{(0)}, \quad v_2^{(0)} = -a_5 \frac{Mz}{z-M} \mu p^{(0)}, \\ S^{(0)} &= a_4 p^{(0)}, \quad H_2^{(0)} = -qa_5 \frac{Mz}{z-M} \mu p^{(0)}, \quad H_1^{(0)} = 0. \end{aligned}$$

Substituting these relations to Eq. (20) and applying (26) we obtain the equality

$$(27) \quad \frac{\mu z (\gamma - 1)(1 - M^2)^2}{2M(2 + (\gamma - 1)M^2)(z^2 - 1)(z - M)} h(z) p^{(0)} = 0,$$

where

$$h(z) = -z^2 + 2Mz + \frac{2}{\gamma - 1}.$$

As  $p^{(0)}$  must not be equal to zero then (27) becomes

$$(28) \quad h(z) = 0.$$

If Eq. (28) has roots  $z$  in the domain of  $z$  described above then system (19), (20) has a nonzero solution  $\mathbf{X}$ . Thus, the condition  $g(\mathcal{M}_0, \eta, \xi, \omega, \lambda) \neq 0$  is equivalent to the condition  $h(z) \neq 0$ . Eq. (28) has the roots

$$z_{1,2} = M \pm \sqrt{M^2 + \frac{2}{\gamma - 1}}.$$

The root  $z_2 < 0$ . The root  $z_1 > 1$  if

$$(29) \quad 2M(\gamma - 1) + 3 - \gamma > 0.$$

It is obvious that (29) holds for  $\gamma < 3$  (see also Gardner and Kruskal, 1964). But inequality (29) is valid also for all  $\gamma (\gamma > 1)$ . Indeed, (29) can be rewritten as

$$2\gamma M^2 - (\gamma - 1) + 2(1 - M)(1 + \gamma M) > 0.$$

By virtue of (7), the last inequality is fulfilled for all  $\gamma (\gamma > 1)$ . So, the root  $z_1 > 1$ , and  $h(z) \neq 0$  for all  $\eta > 0$ ,  $\xi \in R$ , and for  $\lambda$  with  $\text{Re } \lambda > 0$ , where  $\lambda$  is the solution of Eq. (21).

Thus, the boundary conditions (11) of the mixed Problem  $\mathcal{F}$  satisfy the LC. It implies the *linear stability* (see Sect. 1) of fast parallel MHD shock waves in a polytropic gas.

To separate the domain of fulfilment of the ULC (the domain of *structural stability*) it is necessary to find a subdomain of the whole domain of parameters

$$\gamma > 1, \quad 0 < q < M, \quad \frac{\gamma - 1}{2\gamma} < M^2 < 1$$

where  $h(z) \neq 0$  as well for such  $z$  which correspond to the case  $\eta = 0$ ,  $\lambda = \lambda_0$  ( $\text{Re } \lambda_0 = 0$ ).

### 5. The structural stability domain for a fast parallel MHD shock wave in a polytropic gas

To derive a condition for the structural stability (the domain II, see Sect. 1) of fast parallel MHD shock waves it is necessary to carry out a relatively complicated and delicate analysis of Eq. (28) for the case  $\eta = 0$ ,  $\operatorname{Re} \lambda = 0$ . In this connection, we will here simply present this condition, which is derived in the Appendix.

So, as is proved in the Appendix, boundary conditions (11) satisfy the ULC if

$$(30) \quad g(z_1) > 0,$$

where

$$z_1 = M + \sqrt{M^2 + \frac{2}{\gamma - 1}}, \quad g(z) = (zM - 1)z^4 + q^2\{(zM - 1)(z^2 - 2)z^2 - q^2(z^2 - 1)^2\}.$$

Thus, inequality (30) is the condition for the structural stability of fast parallel MHD shock waves in a polytropic gas.

Let us analyse condition (30) a little. It is clear that this condition is fulfilled, for example, for the asymptotic case of a weak magnetic field ( $q \ll 1$ ). Indeed, in such a case inequality (30) is equivalent to the condition  $g_0(z_1) > 0$ , where  $g_0(z) = (zM - 1)z^4$ . In view of (7), the last inequality holds because

$$(31) \quad M + \sqrt{M^2 + \frac{2}{\gamma - 1}} > \frac{1}{M}.$$

Carrying out a more delicate algebraic analysis (we do not present it here) one can show that condition (30) holds also, for example, in the domain

$$M^2 > \frac{1}{2}, \quad \gamma < 3.$$

On the other hand, it is not difficult to see that the asymptotic case of minimal admissible Mach numbers

$$M^2 - \frac{\gamma - 1}{2\gamma} \ll 1$$

(in this case  $z_1$  is close to  $1/M$ ) is an example of the violation of condition (30), i.e., domain III (see Sect. 1) is not empty, unlike the case of gas dynamic shock waves ( $q = 0$ ) in a polytropic gas which are structurally stable in the whole domain of parameters (see Blokhin, 1986).

*Remark 1.* – In the work of Blokhin and Trakhinin (1994) the following *a priori* estimates are obtained for the stability problem of fast MHD shock waves in the asymptotic case of a weak magnetic field ( $q \ll 1$ ):

$$\begin{aligned} \|\mathbf{U}(t)\|_{W_2^2(R_+^2)} &\leq K_1 \|\mathbf{U}_0\|_{W_2^2(R_+^2)}, \quad 0 < t \leq T < \infty, \\ \|F\|_{W_2^3((0,T) \times R)} &\leq K_2, \end{aligned}$$

where  $K_{1,2} > 0$  are constants depending on  $T$ , and it is assumed that the relation between the magnetic field and the normal to the front of a shock wave is arbitrary. So, such *a priori* estimates hold for the solutions of Problem  $\mathcal{F}$  as well if  $q \ll 1$ .

*Remark 2.* – In domain III, where  $g(z_1) \leq 0$ , the question of shock wave stability can be solved only for the quasilinear formulation of the stability problem, i.e., we have to consider the initial quasilinear system (1) and the jump conditions (4).

*Remark 3.* – For the case of an arbitrary state equation  $E = E(\rho, S)$  the equation for determination of roots  $z$  (for a polytropic gas it is Eq. (28)) also does not depend on the magnetic field (see Gardner and Kruskal, 1964) and coincides with the corresponding one in Gas Dynamics. Thus, as was noted by Gardner and Kruskal (1964), the domain of linear stability (the union of domains II and III) is determined by the condition derived first by Erpenbeck (1962). Following the technique described in the Appendix it is possible to separate a subdomain of the domain presented by Erpenbeck's condition where the ULC is fulfilled. Such a subdomain is the domain of structural stability (domain II) of fast parallel MHD shock waves for an arbitrary state equation.

## 6. Conclusions

The stability of fast shock waves in Magnetohydrodynamics for an ideal medium is studied. We considered the case of a polytropic gas, and the magnetic field is assumed to be parallel to the normal to the front of a shock wave.

On the basis of these results we can form the following conclusions. MHD shock waves of the type considered are stable, in the sense that perturbations do not grow with time, in a polytropic gas with arbitrary  $\gamma$ . However, such a linear stability does not always guarantee the real existence of a corresponding curvilinear shock wave as a physical structure. In this connection, a much more important result is that of structural stability. Only in the domain of structural stability, where the magnetic field, the Mach number and the adiabatic exponent  $\gamma$  satisfy condition (30), are we able to conclude the real existence of shock waves of the type considered.

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## APPENDIX

Here we prove that the domain II (see Sect. 1) for stability Problem  $\mathcal{F}$  is determined by condition (30). At the beginning we consider, for simplicity, the case corresponding to gas dynamic shocks. So, we formally set  $q = 0$ . Then

$$\mu = -\frac{\sqrt{z^2 - 1}}{z}.$$

It is clear that only the root  $z = z_1$  ( $z_1 > 1$ ) of Eq. (28) can correspond to the case  $\eta = 0$ ,  $\lambda = \lambda_0 = i\delta$ ,  $\delta \in R$  (see Sect. 4). In the domain  $z > 1$  the function

$$\delta = \delta(z) = \frac{1}{\sqrt{z^2 - 1}}$$

decreases, and the function

$$\xi = \xi(z) = \frac{z - M}{M\sqrt{z^2 - 1}}$$

( $s = i\xi$ ) decreases up to its minimum  $1/M$  and increases for  $z > 1/M$ .

Solving (21) for  $q = 0$  we find

$$\lambda = \lambda_{1,2} = \frac{M^2 s \pm \sqrt{M^2 s^2 + 1 - M^2}}{1 - M^2}.$$

It is easy to see that  $\lambda_0 = \lambda_1|_{\eta=0} = i\delta$ . Then we have

$$(A.1) \quad \delta = \frac{M^2 \xi + \sqrt{M^2 \xi^2 + M^2 - 1}}{1 - M^2}.$$

The graph of the function  $\xi = \xi(z)$  has two points of intersection  $z = r_{1,2}$  ( $1 < r_1 < 1/M < r_2$ ) with the line  $\xi = \hat{\xi} = \text{const}$  for  $z > 1$ ,  $z \neq 1/M$  (the constant  $\hat{\xi}$  is supposed to lie in the range of values of the function  $\xi = \xi(z)$  for  $z > 1$ ). One of these points of intersection corresponds to the case  $\eta = 0$ ,  $\lambda = \lambda_0 = i\delta$ , where  $\delta$  is determined by formula (A.1). In view of (A.1),  $\delta'(\xi) > 0$ . On the other hand,  $\delta'(\xi) = \delta'(z)/\xi'(z)$ . As  $\delta'(z) < 0$  under  $z > 1$  then the interval  $z > 1/M$ , which contains the point of intersection  $z = r_2$  ( $\xi'(r_2) > 0$ ), determines the domain II. But  $z_1 > 1/M$  (see (31)), and hence gas dynamic shock waves ( $q = 0$ ) are structurally stable in a polytropic gas.

Now we proceed to the more complicated case  $q > 0$ . From (24) we have

$$\xi = \xi(z, q) = \xi(z)\eta(z, q)$$

where the function  $\xi(z)$  is described above, and

$$\eta = \eta(z, q) = \sqrt{\frac{z^2(1+q^2) - q^2}{z^2 - q^2}}.$$

It is not difficult to check that the function  $\eta$  (as a function of  $z$ ) decreases in the interval  $z > 1$ . Hence the function  $\xi = \xi(z, q)$  for  $z > 1$  decreases up to its minimum  $z_*$  ( $z_* > 1/M$ ) and increases for  $z > z_*$ .

In view of the continuous dependence of  $\eta$  on the parameter  $q$ , the point of intersection of the graph of the function  $\xi = \xi(z, q)$  with the line  $\xi = \hat{\xi} = \text{const}$ , which corresponds to the case of no roots with  $\eta = 0$ ,  $\lambda = \lambda_0 = i\delta$ , lies to the right of  $z_*$ , for sufficiently small  $q$  ( $z_*$  is close to  $1/M$ ), and cannot jump over the interval  $1 < z < z_*$  while  $q$  increases up to  $M$  ( $0 < q < M$ ). Therefore the domain determined by the condition  $z_1 > z_*$  presents domain II.

To find  $z_*$  we have to solve the equation  $\xi'_z(z, q) = 0$  ( $z > 1$ ) that is equivalent to the following equation

$$(A.2) \quad g(z) = (zM - 1)z^4 + q^2\{(zM - 1)(z^2 - 2)z^2 - q^2(z^2 - 1)^2\} = 0.$$

The coefficient of  $z^5$  in (A.2) is positive. Eq. (A.2) has the root  $z = z_*$ , and other roots are either less than  $1$  ( $1 < z_*$ ) or complex. Hence the polynomial  $g(z)$  is positive on the interval  $z > z_*$  and negative for  $1 < z < z_*$ . Consequently, the inequality  $z_1 > z_*$  is equivalent to the inequality

$$g(z_1) > 0.$$

Therefore condition (30) describes domain II for the stability Problem  $\mathcal{F}$ .

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